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ABSTRACT

The limitations of the ledger method in writing formal proofs are discussed. Details are given of a flow-proof method, with an attempt made to describe how to deal with most special situations involving the structuring of proofs. Nine examples of flow proofs in gemoetry are included. (DT)

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AN INTRODUCTION TO FLOW PROOF

A Manual For Teachers

1969-1970

SE 015 931

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Notation

For purposes of clarification, some standard notation should be agreed upon. Throughout this paper, the following notation will be utilized. It is not the writer's contention that this notation is the best, nor even the most efficient. Rather, it is used in order that a common basis for discussion can be reached.

$\overleftrightarrow{AB}$ the line passing through points A and B (a set of points).

$\overline{AB}$ the segment whose endpoints are A and B (a set of points).

$\overrightarrow{AB}$ the ray with endpoint A which passes through point B (a set of points).

AB the length or measure of $\overline{AB}$ (a number).

$\angle ABC$ the angle whose rays are $\overrightarrow{BA}$ and $\overrightarrow{BC}$ (a set of points). When no possibility for confusion exists, a single letter may be used.

$m\angle ABC$ the measure (usually degree measure) of $\angle ABC$ (a number).

$=$ names the same thing as (used with a set of points or a number).

$\cong$ can be made to coincide with (used with sets of points). (Also used to indicate a specific 1-1 correspondence).

The SSS, SAS, ASA, AAS, HL and HA postulates and theorems will be based upon congruences of segments and angles rather than equality.

Limitations of the Ledger Method

Accurate evaluation of proofs submitted by high school geometry students presents a common problem shared by their teachers. In evaluating proofs, the teacher

wants to give a student all the credit that he deserves. However, the teacher would be remiss in his duties if, because of inadequate marking procedures, he allowed the student continually to submit proofs of poor quality. Thus, it is important that teachers be able to recognize whether a student's written work actually demonstrates an understanding of a particular problem and the nature of proof in general.

The ledger or two-column method of structuring proofs, which is illustrated in nearly all geometry textbooks and used by most teachers in their classrooms, does not always allow the student to represent adequately the relationships among statements in the proof. Consequently, it is often difficult for the teacher to determine from a student's work how much the student really understands. Moreover, when a student is having trouble, it is hard for the teacher to locate the source of his difficulty.

Proofs involving triangle congruence are particularly difficult to evaluate because some students, not always the poorer ones, seem to proceed along the following plan:*

1. Write part of the given, usually the first statement listed in the given, as Statement 1 in the proof.

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*An alternative method used by some students, but frowned upon by many teachers, is to list <u>all</u> given statements first, and then to draw conclusions.

2. List all (if any) possible conclusions which can be drawn from Statement 1 and known theorems and postulates.

3. Write another statement in the given as the next statement in the proof.

4. List all possible conclusions which can be drawn from the statements in the proof and known theorems and postulates.

5. Continue the above process until all statements in the given are listed and all conclusions are drawn.

6. Find statements (usually three) in the proof which will establish the desired congruence.

Some students seem to function quite effectively using this technique. Although they occasionally have unnecessary statements in their proofs, it can be conceded that they really do have a good, basic understanding of proof and can apply this understanding in the solution of most problems. But, there are students lacking a basic understanding of proof who are actually able to conceal (whether intentionally or not) their difficulty by using a technique which is quite similar to that described above. These students are able to complete Steps 1 through 5 in the plan, but they sometimes have difficulty in establishing the desired congruence. Rather than throw in the towel at this point, one of these students will select a theorem or postulate which <u>seems</u> as if it might be appropriate, e.g. ASA, and reason that "Surely somewhere in all those statements I've listed are two pairs of angles and a pair of sides which will work." He then

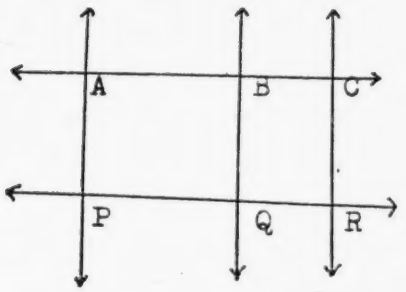
states the desired congruence and lists ASA as his reason. If the student is lucky, and ASA is the correct reason, the teacher is likely to give him full credit for his efforts. If AAS is the correct reason, the teacher is faced with a dilemma. Was the student way off base or was he simply guilty of a careless error and, therefore, deserving of some form of partial credit?

Development of Flow-Proof Method

In attempts to counteract this kind of dilemma, and certain other defects of the traditional method, many teachers have devised alternative methods for structuring proofs. Most of these have been modifications of the ledger method. In recent years, however, the markedly different flow-proof method of structuring proofs has been developed. Although several teachers have devised similar methods, the most highly-developed technique appears to be that of Frank Allen, past-president of the NCTM and formerly chairman of the mathematics department at Lyons Township High School, La Grange, Illinois. Allen, basing the idea for his method on techniques used in Allendoerfer and Oakley's Principles of Mathematics (1954), first used the term "flow proof." He has also used the terms flow-diagram proofs and programmed proofs. This method has been used successfully by an increasing number of teachers.

To illustrate the flow-proof method, consider its application in the following proof:

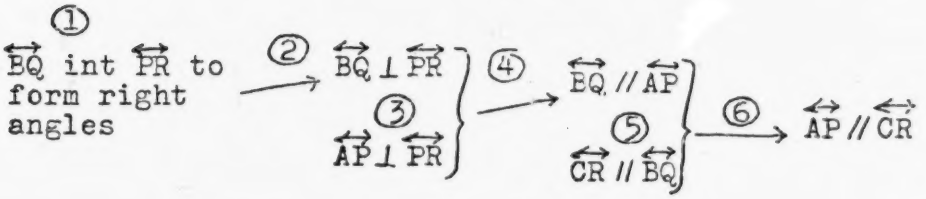
EXAMPLE 1



Given: The plane figure at the left with $\overleftrightarrow{AP} \perp \overleftrightarrow{PR}$, $\overleftrightarrow{BQ}$ intersects $\overleftrightarrow{PR}$ to form right angles with it, $\overleftrightarrow{CR} \parallel \overleftrightarrow{BQ}$.

Prove: $\overleftrightarrow{AP} \parallel \overleftrightarrow{CR}$.

Proof


① $\overleftrightarrow{BQ}$ int $\overleftrightarrow{PR}$ to form right angles
② $\overleftrightarrow{BQ} \perp \overleftrightarrow{PR}$
③ $\overleftrightarrow{AP} \perp \overleftrightarrow{PR}$
④ $\overleftrightarrow{BQ} \parallel \overleftrightarrow{AP}$
⑤ $\overleftrightarrow{CR} \parallel \overleftrightarrow{BQ}$
⑥ $\overleftrightarrow{AP} \parallel \overleftrightarrow{CR}$

1. Given.
2. If two lines meet to form right angles, then they are perpendicular.
3. Given.
4. If two lines are perpendicular to the same line, then they are parallel.
5. Given.
6. If two lines are parallel to the same line, then they are parallel.

Attention should be drawn to several aspects of this proof:

a. The number of the initial statement in each line is written above the statement (e.g. Statements 1, 3, and 5).

b. Each implication is numbered, with the number above the arrow. (In a ledger proof, Statement 4 would be $\overleftrightarrow{BQ} \parallel \overleftrightarrow{AP}$).

c. The brace coupling Statements 2 and 3 is needed because Implication 4 depends upon both of those statements.

d. No brace is needed after Statement 1 because Statement 2 follows directly from a single statement.

e. The reasons are listed, as compactly as possible, after all statements are written. This allows the student to complete his train of thought without interruptions for reasons.

f. The reason for an implication is written in the if-then form. This is an extremely important part of the method because it stresses the deductive nature of proof.

Allen first used this method with algebraic proofs, many of which do not require braces (See Example 2). However, nearly all geometric proofs of any consequence will require the use of braces.

EXAMPLE 2

Given: $5x + 2 = 22$ Proof

Prove: $x = 4$ $5x + 2 = 22 \overset{①}{\longrightarrow} 5x = 20 \overset{②}{\longrightarrow} x = 4$ $\overset{③}{\longrightarrow}$

1. Given. 2. If a=b then a+c = b+c. 3. If a=b, then ac=bc.

Proofs Involving Triangle Congruence

In particular, proofs involving triangle congruence will always have a brace combining at least three statements. Three statements will be combined whenever SAS, ASA, AAS, or SSS (Example 3) is used as a reason and four statements will be combined whenever HA or HL (Example 4) is used as a reason.

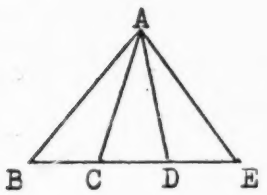
A Common Problem

Initially, many students have difficulty with the flow-proof structure because they don't plan ahead. Statements which should be aligned vertically are not, making it difficult to include them in braces (See Example 5). The students probably should not be penalized for this mistake. Rather, this should serve to call their attention to the importance of careful planning in the future. If past experience is any indication, this problem will vanish by the end of the second week of the study of proof.

Lengthy Proofs

Frequently, the length of a proof will make it difficult to complete the proof on one horizontal level. Proofs may be completed in two or more levels (See Example 6), or lengthwise on the paper (See Example 7).

EXAMPLE 3



Given: The plane figure at the left with $\overline{AC} \cong \overline{AD}$, $\overline{BC} \cong \overline{CD} \cong \overline{DE}$, $\angle ABC \cong \angle AED$.

Prove: $\angle BAC \cong \angle EAD$.

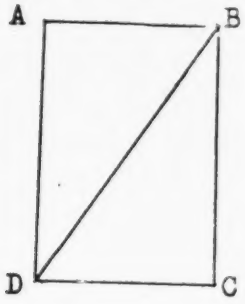
Proof

$$\angle ABC \cong \angle AED \xrightarrow{①} \left.\begin{array}{l} \overline{AB} \cong \overline{AE} \\ \overline{BC} \cong \overline{DE} \\ \overline{AC} \cong \overline{AD} \end{array}\right\} \xrightarrow{②,③,④} \triangle ABC \cong \triangle AED \xrightarrow{⑤} \angle BAC \cong \angle EAD$$

1. Given.
2. If two angles of a triangle are congruent, the sides opposite them are congruent.
3. Given.
4. Given.
5. If the sides of one triangle are congruent, respectively, to the sides of another triangle, the triangles are congruent.
6. If two angles are corresponding parts of congruent triangles, they are congruent.

Note that reasons for implications are stated in if-then form.

EXAMPLE 4



Given: The plane figure at the left with $\angle A$ a right angle, $\overline{AD} \cong \overline{CB}$, $\overleftrightarrow{DC} \perp \overleftrightarrow{CB}$.

Prove: $\triangle ABD \cong \triangle CDB$.

Proof

$\overleftrightarrow{DC} \perp \overleftrightarrow{CB}$ ① → ② $\angle C$ is a rt$\angle$ → ③ $\triangle CDB$ is a rt$\triangle$

④ $\angle A$ is a rt$\angle$ → ⑤ $\triangle ABD$ is a rt$\triangle$

⑥ $\overline{AD} \cong \overline{CB}$

⑦ $\overline{BD} \cong \overline{DB}$

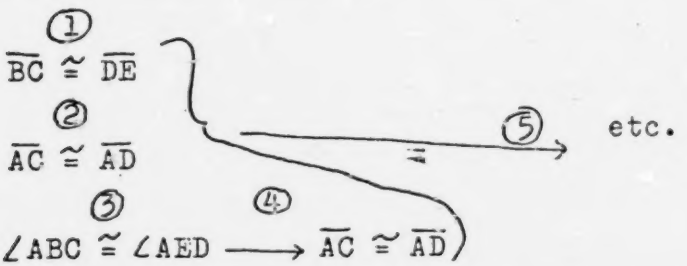
→ ⑧ $\triangle ABD \cong \triangle CDB$

1. Given.
2. If two lines are $\perp$, they meet to form rt $\angle$'s.
3. If a triangle contains a rt$\angle$, it is a rt $\triangle$.
4. Given
5. If a $\triangle$ contains a rt$\angle$, it is a rt $\triangle$.
6. Given.
7. Reflexive property of congruence.
8. If the hypotenuse and one leg of a right $\triangle$ are congruent, respectively, to the hypotenuse and leg of another right $\triangle$, the triangles are congruent.

Note that reasons for implications are stated in if-then form.

EXAMPLE 5

AN INCORRECT PROOF(PARTIAL) OF EXAMPLE 3



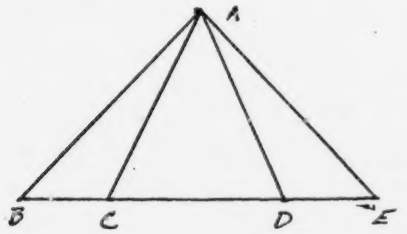
Omitting Reasons

Example 6 also illustrates a convenient, time-saving device which can be used by teachers. Because of the structure involved in a flow proof, a teacher can, by simply examining the statements in a proof, evaluate student work with a high degree of accuracy. He can also save class time by instructing students to omit reasons on quizzes and in-class exercises.

Two or More Implications From One Statement

Example 7 also illustrates a convenient way for diagramming a proof in which two or more implications follow from the same statement. In Example 7, both Statements 5 and 7 follow directly from Statement 4. This is shown by drawing two arrows from Statement 4. Occasionally, this procedure cannot be followed, as will be seen in the next section.

EXAMPLE 6*



Given: The plane figure at the left with $\overline{BD} \cong \overline{CE}$, $\angle B \cong \angle E$, $\angle ABC \cong \angle AED$.

Prove: $\triangle ABC \cong \triangle AED$.

Proof

① C is between B and D on $\overrightarrow{BE}$ ⟶ ② $BC + CD = BD$

③ D is between C and E on $\overrightarrow{BE}$ ⟶ ④ $CD + DE = CE$

⑤ $\overline{BD} \cong \overline{CE}$ ⟶ ⑥ $BD = CE$

②, ④, ⑥ ⟶ ⑦ $BC + CD = CD + DE$ ⟶ ⑧ $BC = DE$ ⟶ ⑨ $\overline{BC} \cong \overline{DE}$

⑩ $\angle B \cong \angle E$

⑪ $\angle ABC \cong \angle AED$

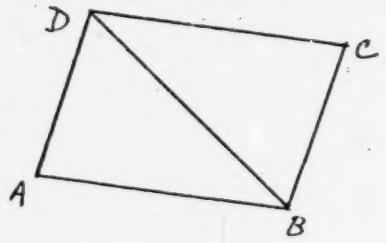
⑨, ⑩, ⑪ ⟶ ⑫ $\triangle ABC \cong \triangle AED$

*The reasons in this proof were intentionally omitted.

EXAMPLE 7

Given: The plane figure at the right with $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$.

Prove: ABCD is a parallelogram.



Proof

$\left.\begin{array}{l} \overline{AB} \cong \overline{CD} \ (1) \\ \overline{AD} \cong \overline{CB} \ (2) \\ \overline{BD} \cong \overline{DB} \ (3) \end{array}\right\} \xrightarrow{(4)} \triangle ABD \cong \triangle CDB \begin{cases} \xrightarrow{(5)} \angle ABD \cong \angle CDB \xrightarrow{(6)} \overline{AB} \parallel \overline{CD} \\ \xrightarrow{(7)} \angle ADB \cong \angle CBD \xrightarrow{(8)} \overline{AD} \parallel \overline{BC} \end{cases} \xrightarrow{(9)}$ ABCD is a parallelogram.

1. Given.
2. Given.
3. Reflexive property of congruence.
4. If 3 sides of one $\triangle$ are $\cong$, respectively, to 3 sides of another $\triangle$, the triangles are $\cong$.
5. If two angles are corresponding parts of congruent triangles, the angles are congruent.
6. If two segments are cut by a transversal so that the alternate interior angles are congruent, the segments are parallel.
7. Same as Reason 5.
8. Same as Reason 6.
9. If the opposite sides of a quadrilateral are parallel, it is a parallelogram.

Numbering Repeated Statements

Sometimes one statement will be the antecedent in one implication and a partial antecedent in another implication. In Example 8, the statement $\overline{AB} \cong \overline{AD}$ is used to imply two others. If $\overline{AB} \cong \overline{AD}$ were listed only once, it would be difficult to structure the proof. Consequently, it is listed twice in the proof. When this is done, it is best to use two different numbers for the statement.

Indirect Proof

In an article in the Mathematics Teacher, Donn Klingler,* formerly a colleague of Frank Allen at Lyons Township High School, describes a method for flow-diagraming an indirect proof. Example 9 illustrates how a slightly modified version of Klingler's method can be applied.

Incomplete Proofs

Occasionally, students will not be able to complete a proof. When this occurs, they should be enccuraged to write as much of the proof as they can, leaving spaces for details which cannot be completed. This procedure serves a dual purpose:

1. The teacher will be able to pinpoint areas of difficulty and offer assistance. Also, some

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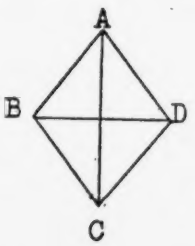
*Donn L. Klingler. "Structuring a Proof." The Mathematics Teacher, 57 (April, 1964) 201.

EXAMPLE 8



Given: The plane figure at the left with $\overline{AB} \cong \overline{AD}$, $\angle ABC \cong \angle ADC$.

Prove: $\triangle ABC \cong \triangle ADC$.

Proof

① $\overrightarrow{BD}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ ② $\longrightarrow m\angle ABD + m\angle DBC = m\angle ABC$

③ $\overrightarrow{DB}$ is between $\overrightarrow{DA}$ and $\overrightarrow{DC}$ ④ $\longrightarrow m\angle ADB + m\angle BDC = m\angle ADC$

⑤ $\angle ABC \cong \angle ADC$ ⑥ $\longrightarrow m\angle ABC = m\angle ADC$

① ② ③ ④ ⑤ ⑥ $\longrightarrow$ ⑦

⑦ $\longrightarrow m\angle ABD + m\angle DBC = m\angle ADB + m\angle BDC$

⑧ $\overline{AB} \cong \overline{AD}$ ⑨ $\longrightarrow \angle ABD \cong \angle ADB$ ⑩ $\longrightarrow m\angle ABD = m\angle ADB$

⑦ ⑧ ⑨ ⑩ $\longrightarrow$ ⑪ $\longrightarrow m\angle DBC = m\angle BDC$ ⑫ $\longrightarrow$

⑫ $\longrightarrow \angle DBC \cong \angle BDC$ ⑬ $\longrightarrow \overline{BC} \cong \overline{DC}$

⑭ $\overline{AB} \cong \overline{AD}$

⑮ $\overline{AC} \cong \overline{AC}$

⑬ ⑭ ⑮ $\longrightarrow$ ⑯ $\longrightarrow \triangle ABC \cong \triangle ADC$

(Reasons on next page)

EXAMPLE 8 (continued)

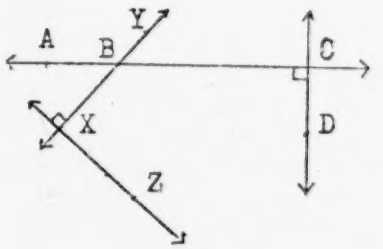
1.	Given, by diagram.	10.	Same as Reason 6.
2.	If $\overrightarrow{AC}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AD}$, then $m\angle BAC + m\angle CAD = m\angle BAD$.	11.	If $a + b = c + d$ and $a = c$, then $b = d$.
3.	Given, by diagram.	12.	If the measures of two angles are equal, the angles are congruent.
4.	Same as Reason 2.	13.	If two angles of a triangle are congruent, the sides opposite them are congruent.
5.	Given.	14.	Given.
6.	If two angles are congruent, their measures are equal.	15.	Reflexive property of congruence.
7.	If $a = b$, $c = d$ and $b = d$, then $a = c$.	16.	If the sides of one triangle are congruent, respectively, to the corresponding sides of another triangle, the triangles are congruent.
8.	Given.		
9.	If two sides of a triangle are congruent, the angles opposite them are congruent.		

Note that reasons for all implications are stated in if-then form.

EXAMPLE 9

Given: The plane figure at the right with $\overleftrightarrow{XY} \perp \overleftrightarrow{XZ}$, $\overleftrightarrow{AC} \perp \overleftrightarrow{CD}$, $\overleftrightarrow{AC}$ intersects $\overleftrightarrow{XY}$.

Prove: $\overleftrightarrow{XZ}$ intersects $\overleftrightarrow{CD}$.



Proof

$\left.\begin{array}{l} \overleftrightarrow{XZ} \parallel \overleftrightarrow{CD} \ (1) \\ \overleftrightarrow{XY} \perp \overleftrightarrow{XZ} \ (2) \end{array}\right\} \xrightarrow{(3)} \overleftrightarrow{XY} \perp \overleftrightarrow{CD}$

$\left.\begin{array}{l} \overleftrightarrow{XY} \perp \overleftrightarrow{CD} \\ \overleftrightarrow{AC} \perp \overleftrightarrow{CD} \ (4) \end{array}\right\} \xrightarrow{(5)} \overleftrightarrow{XY} \parallel \overleftrightarrow{AC}$

$\left.\begin{array}{l} \overleftrightarrow{XY} \parallel \overleftrightarrow{AC} \\ \overleftrightarrow{AC} \text{ int } \overleftrightarrow{XY} \ (6) \end{array}\right\} \xrightarrow{(7)}$ It is false that $\overleftrightarrow{XZ} \parallel \overleftrightarrow{CD}$ $\xrightarrow{(8)} \overleftrightarrow{XZ}$ int $\overleftrightarrow{CD}$

1. Assumption.
2. Given.
3. If a line is perpendicular to one of two parallel lines, it is perpendicular to the other.
4. Given.
5. If two lines are perpendicular to the same line, they are parallel.
6. Given.
7. If an assumption leads to contradictory statements, it is false.
8. If a statement is false, its contradiction is true.

Note that reasons for implications are stated in if-then form.

teachers like to give partial credit.

2. After seeing the incomplete proof written out, the student may be able to determine how to complete it.

Other Situations

It was indicated earlier that an attempt has been made to describe how to deal with most special situations involving structuring proofs. If a situation which has not been discussed arises, please feel free to contact the author.